"A Study of Differentiation and Integration Operators of Fractional Order and their Applications "

A
Synopsis

Submitted in

Partial fulfillment for the degree

of

Doctor of Philosophy

(MATHEMATICS)

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March-2023

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1. TITLE OF THE THESIS:-

"A Study of Differentiation and Integration Operators of Fractional Order and their Applications "

2. INTRODUCTION:-

Fractional calculus (A calculus of arbitrary order) is a generalization of classical calculus. It is a wide area of research in the field of physical science. It extends the derivative of an integer order to an arbitrary order is a natural generalization of classical calculus. The concept of fractional calculus (fractional derivative and fractional integral) is not new because fractional calculus is the part of special functions , Special functions is a real or complex valued functions of one or more real or complex variables which is specified so completely that its numerical values could in principle be tabulated [12].

Although Fractional calculus originated in the $17th\,century$ but The development of Fractional calculus gained momentum after 1974 when University of New Haven organized first international conference on fractional calculus in 1974. Since then it has well-known branch of mathematics and researchers contributing to find applications in this area.

A calculus of arbitrary order i.e. Fractional calculus and its applications have the significant role in the progress of science and engineering specifically in the field of special functions like Mittag-Leffler function , generalized Mittag-Leffler function generalized M-Series, k_2 function , k_4 function, Miller-Ross function , Wright's function and more generally by the Meijer's G-Function, Fox's H-function , Saxena's I-function and Sudland , Baumann and Nonnenmacher's $\varkappa$-aleph function. Last five year applied mathematicians and physicists obtained the fractional calculus operators to be very useful in a variety of fields such as quantitative biology, electro chemistry, scattering theory , diffusion theory, signal processing, image processing, kinetic theory, heat transfer theory, circuit theory, transport theory, probability and potential theory etc. Fractional calculus plays the most significant role in dealing with the practical applications of special functions solving in various problems in fluid mechanics, heat conduction, quantum

mechanism, kinetic and diffusion equations. The main purpose of this research endower is to study extensively theory of fractional calculus and its existing applications. New definitions of fractional derivative and integral is also under consideration in establishing new results. An attempt will be made to be modify/extend some existing results in theory as well as in applications. Some novel results will be obtained to enrich the theory of fractional calculus. As well as new applications will be generated in different fields of science and engineering.

2.1 INTRODUCTION OF FRACTIONAL CALCULUS:-

Fractional calculus (A calculus of arbitrary order which may be real or complex) is a generalization of ordinary differentiation of non-integer case. The credit of developing fractional calculus goes to an L 'Hospital and Leibnitz. In 30th September 1695 L 'Hospital asked the question as to the meaning of $d^n y/dx^n$ if n=1/2; that is "what if n is fractional". He replied "An apparent paradox, from which one day useful consequences will be drawn" [2, 33]. These were the precious words of Leibnitz became true after 300 year and the fractional calculus was originated. After that several mathematicians namely Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov, Weierstrass, Heaviside, Mittag-Leffler, Erdelyi and Riesz etc. have made significant contributions in the field of fractional calculus. Fractional calculus may be considered an old and yet novel topic of research. The nineteenth century is considered as the age of fractional calculus because many renowned mathematicians developed this theory in this century. The first major study of fractional calculus began in 1832 when Liouville [28] considered $d^{\frac{1}{2}}/dx^{\frac{1}{2}}(e^{2x})$ and solved some problems in mechanics and geometry by the use of fractional calculus operators. In the twentieth century credit of growth of fractional calculus goes to Bertram Ross who organized the first international conference on "Fractional Calculus and Its Applications" at the University of New Haven in June 1974 and published all presented research papers as its proceedings [11].

2.1.1 DEFINITION AND PROPERTIES OF FRACTIONAL CALCULUS:-

We consider first logical definition of fractional derivative which was given by Riemann-Liouville. After that several mathematicians attempted to define forms of fractional integral and derivative. Some definitions of the fractional calculus were given by Riemann-Liouville left

and right fractional integral and derivative operators, Weyl fractional derivative and integral operators and Caputo fractional operators, Taun and Saigo's operators, Grunwald – Letnikov differential operators, which are given as:-

1. Riemann-Liouville left and right sided fractional derivative [33, 5, 22]:-

The Riemann-Liouville left-sided fractional derivative of order α

$$ {}_{\rho}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-a)} \left(\left(\frac{d}{dx} \right)^{n} \int_{\rho}^{x} \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \right), \quad (n = [\alpha] + 1) $$

The Riemann-Liouville right-sided fractional derivative of order α

$$ {}_{x}D_{\lambda}^{\alpha} f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \left((\frac{d}{dx})^{n} \int_{x}^{\lambda} \frac{f(t)}{(t-x)^{\alpha-n+1}} dt \right), (n = [\alpha] + 1) $$

Where the order α in $n-1 \leq \alpha < n$ and Γ represent the Euler's gamma function

Where $[\alpha]$ denotes the integral part of α

2. Riemann-Liouville left and right sided fractional integrals [33, 5, 22]:-

The Riemann-Liouville left-sided fractional integrals of order α

Let $f(x) \in L(\rho, \lambda), \alpha \in c, Re(\alpha) > 0.$

Then $ {}_{\rho}I_{x}^{\alpha} f(x) = {}_{\rho}D_{x}^{-\alpha} f(x) = I_{\rho+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \left(\int_{\rho}^{x} \frac{f(t)}{(x-t)^{1-\alpha}} dt \right), x > \rho $

The Riemann-Liouville right-sided fractional integrals of order α

Let $f(x) \in L(\rho, \lambda), \alpha \in c, Re(\alpha) > 0.$

Then $ {}_{x}I_{\lambda}^{\alpha} f(x) = {}_{x}D_{\lambda}^{-\alpha} f(x) = I_{\lambda-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \left(\int_{x}^{\lambda} \frac{f(t)}{(x-t)^{1-\alpha}} dt \right), x < \lambda $

3. Modified Riemann-Liouville fractional derivative [19]:-

The modified Riemann-Liouville fractional derivative of order $\alpha, \alpha < 0,$ is defined by the expression

$$f^{\alpha}(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z (z-\xi)^{-\alpha-1} f(\xi) d\xi$$

For positive α then $(f^{(\alpha-1)}(x))'$, $0< \alpha < 1$

$$f^{\alpha}(z) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dz} \int_0^x (z-\xi)^{\alpha-1} (f(\xi) - f(0))\, d\xi \right), (0 < \alpha < 1)$$

And $f^{\alpha}(z) = (f^{(\alpha-n)}(z))^{(n)}$, $n \leq \alpha < n+1, n \geq 1$

4. Grunwald-Letnikov fractional derivative [13, 33]:-

The a^{th} order fractional derivative of a function denoted by $D^{\alpha} f$

$$D^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{m=0}^{\frac{t}{h}} (-1)^m \binom{a}{m} f(t - mh)$$

Expanding the binomial coefficient knows

$$\binom{a}{m} = \frac{a!}{m!\,(n-m)!}, \qquad \binom{a}{m} = \frac{\Gamma a + 1}{m!\,\Gamma(a+1-m)}$$

The Grunwald-Letnikov derivative for real numbers

$$D^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{m=0}^{\frac{t}{h}} (-1)^m \frac{\Gamma a + 1}{m!\,\Gamma(a+1-m)} f(t - mh), \;\; a \in R, \qquad a \neq -N$$

5. Caputo fractional derivative [42]:-

The Caputo fractional derivative of order $\alpha > 0$ is introduced by caputo in the form if

$(m-1 < \alpha \leq m), \qquad Re(\alpha) > 0, m \in n$

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\,, \qquad 0 \leq n-1 < \alpha < n$$

$$= \frac{d^n}{dt^n f(t)} \quad , \qquad if\ \alpha = m$$

Where $\frac{d^m f(t)}{dt^m}$ is the m-th derivative of order m of the function f(t) with respect to t.

Where $\lim_{\alpha \to n} \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau = d^n / dt^n f(t)$

6. The Weyl Fractional integral [8, 33]:-

The weyl fractional integral of $f(z)$ of order α , defined by

$$ {}_x W_\infty^\alpha f(z) = {}_z I_\infty^\alpha f(z) = \frac{1}{\Gamma\alpha} \int_0^\infty (t-z)^{\alpha-1} f(t) dt, -\infty < z < \infty$$

Where $\alpha \in C, Re(\alpha) > 0, (-\infty < z < \infty)$

7. The Weyl Fractional derivative [8, 33]:-

The weyl fractional derivative of order α defined by

$$ {}_x D_\infty^\alpha f(z) = D_-^\alpha f(z) = (-1)^n (\frac{d}{dz})^n {}_z W_\infty^{n-\alpha} f(z)$$

$$= (-1)^n (\frac{d}{dz})^n \frac{1}{\Gamma(n-\alpha)} \int_x^\infty \frac{f(t)}{(t-z)^{1+\alpha-n}} dt \ , (-\infty < z < \infty)$$

Where $\alpha \in C, n \in N, n-1 < \alpha < n$

8. New definition of fractional derivative given by Khalil [66]:-

Author given a function $f : [0,\infty) \to \infty$ then the conformable fractional derivative of f of order a is defined by

$$T_a f(t) = \lim_{\in \to 0} \frac{f(t + \in t^{1-a}) - f(t)}{\in}$$

For all t>0,$a \in (0,1)$ if f is $a-$ differentiable in some (0,b), b>0 and $\lim_{t \to 0^+} f^{(a)}(t)$ exists,

Then $$f^{(a)}(0) = \lim_{t \to 0^+} f^{(a)}(t)$$

And author define $a \in (n, n+1)$ and f be an n-differentiable at t, where t>0.

Then the conformable fractional derivative of f of order a is defined as

$$T_a(f)(t) = \lim_{\in \to 0} \frac{f^{([a]-1)}\left(t + \in t^{[a]-a}\right) - f^{([a]-1)}(t)}{\in}$$

Where $[\alpha]$ is the smallest integer greater than or equal to α.

9. New definition of fractional derivative given by Caputo [43]:-

Fractional time derivative

$$D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t f(p) \exp[\frac{-\alpha(t-p)}{1-\alpha}]\, dp$$

Where M(α) is a normalization function such that M(0)=M(1)=1, the new fractional derivative is zero when f(t) is constant, as in the usual Caputo fractional time derivative, but contrary to the usual Caputo fractional time derivative, the kernel does not have singularity for t=p

The new fractional time derivative can also be applied to functions do not belong to $H^1(a,b)$ this definition can be formulated also for $f \in L^1(-\infty, b)$ and for any $\alpha \in [0,1]$ as

$$D_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_{-\infty}^{t} \big(f(t) - f(P)\big) \exp[\frac{-\alpha(t-p)}{1-\alpha}]\, dp$$

10. Tuan and Saigo fractional operators [76]:-

Tuan and Saigo sometimes write $X_+^{\alpha,m}, X_-^{\alpha,m}$ in place of $X_+^{\alpha}, X_-^{\alpha}$ to underline the dimension of the fractional operators. It is obvious that

$$X_+^{\alpha,1} y(x) = I_{0+}^{\alpha} y(x), \qquad X_-^{\alpha,1} y(x) = I_{0-}^{\alpha} x^{-\alpha} y(x)$$

Where $I_{0+}^{\alpha} y(x) = \frac{1}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1}\, y(t) dt$

and $I_{0+}^{\alpha} y(x) = \frac{1}{\Gamma\alpha} \int_x^{\infty} (t-x)^{\alpha-1}\, y(t) dt$

2.2.2 APPLICATIONS OF FRACTIONAL CALCULUS:-

A calculus of arbitrary order became popular due to its applications ; the first credit of applications of fractional calculus goes to Abel [50] who employed it in the solution of an integral equation which emerged in the formulation of the tautochrone problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of bead placed on the wire to the lowest point of the wire is the same regardless of position of the bead on the wire [44]. Fractional calculus and its applications is undergoing rapid developments with more and more convincing applications in the real world. Applications of fractional calculus in many areas like fluid mechanics, theory of viscoelasticity etc. [41]. We witnessed applications of fractional calculus to the solution of time dependent, viscous-diffusion fluid mechanics problems [75]. Also, applications of fractional calculus in the theory of viscoelasticity in which it efforts possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters [16]. Fractional order controller applied to the path-tracking problem in an autonomous electric vehicle [27]. Fractional calculus applied to the sound waves propagation in rigid porous materials in which the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional power of frequency suggests the fact that time derivatives of fractional order might describe the behavior of sound waves in this kind of materials, including relaxation and frequency dependence [78]. Fractional calculus applied in modeling of speech signals [34] and modeling the cardiac tissue electrode interface [64]. Fractional calculus is used to describe the viscous interactions between fluid and solid structure [48]. Fractional differentiation is also applying for edge detection [10]. Wave propagation in viscoelastic horns using a fractional calculus rheology model in which propagation of infinitesimal waves in one dimensional horns with a small cross sectional area change along the longitudinal axis are examined [72].

3. A BRIEF REVIEW OF WORK ALREADY DONE:-

Fractional calculus is an interesting research field as the real life processes can often be represented in a better way by fractional orders of the derivative than by integer order. The fractional calculus is the most popular area of research due to various applications in science, engineering and

statistics. Several renowned mathematicians contributed to this theory over the years, among them Euler, Liouville, Riemann, Weyl, Abel, Lacroix, Fourier, Leibniz, Grunwald and Letnikov. In 1819 Lacroix [11,69] became the first mathematician to publish a paper that mentioned a fractional derivative and found the n^{th} derivative

$$\frac{d^n y}{dx^n} = \frac{p!}{(p-n)!} x^{p-n}, p \geq n$$

The fractional calculus may be considered an old and yet novel topic. It is an old topic since, starting from some speculations of G.W.Leibniz in 1695, 1697 and L.Euler in 1730. It has been developed up to nowadays. A list of mathematicians, who have provided imported contributions up to the middle of our century, Include P. S. Laplace in 1812, J. B. J. Fourier in 1822, N.H. Abel in 1823-1826, J. Liouville in 1832-1873, B. Riemann in 1847, H. Holmgren in 1865-1867, A. K. Grunwald in 1867-1872, A.V. Letnikov in 1868-1872, H. Laurent in 1884, P.A. Nekrassov in 1888, A. Krug in 1890, J. Hadamard in 1892, O.Heaviside in 1892-1942, S.Pincherle in 1902, G.H. Hardy and J.E.Littlewood in 1917-1928, H. Weyl in 1917, P.Levy in 1923, A. Marchaud in 1927, H.T. Davis in 1924-1936, A. Zygmund in 1935-1965, H. Kobar in 1940, D. V. Widder in 1941, M. Riesz in 1949.

In 1972, Love [15] obtained a pair of index laws for fractional integrals and derivatives. After that Bertram Ross who organized the first international conference on fractional calculus and its applications at the University of New Haven in June 1974.

In 1984, Raina [62] obtained fractional derivative of general system of polynomial. In 1985, Al-Bassam [36] studied the fractional calculus method in solving some classes of differential equations of Hermit's type and given an applications of fractional calculus to differential equation.

Nonnenmacher [70] represented the fractional integral and differential equation and solution of Levy-type Probability densities in 1990.

Samko, Kilbas and Marichev [68] by publishing a book on fractional integrals and derivatives: Theory and Applications in 1993 and 20^{th} century a great contributions was made by Miller and Ross [33] by publishing a book on fractional calculus and fractional differential equation.

Atakishiyev,Vicent, Kurt and Wolf [49] have given the continuous vs. discrete fractional Fourier transforms in 1999.

In 2000, Kiryakova [73] provided the multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus, after that Kilbas and Trujillo [3] defined the differential equations of fractional order: methods results and problem-1 in 2001.

In 2002, Saxena, Mathai and Haubold [58] have given new research results on fractional kinetic equations.

In 2003, Debnath [35] defined the recent applications of fractional calculus to science and engineering. After that Loverro [2] has given the fractional calculus: history, definitions and applications for the engineer in 2004.

Mainardi, Pagnini and Saxena [17] solved the Fox H functions in fractional diffusion in 2005.

In 2006, Jumarie [19] obtained the modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results. Agarwal [52] defined Fractional variational calculus in terms of Riesz fractional derivatives in 2007.

Kilbas and Sebastian [4] obtained the generalized fractional integration of Bessel function of the first kind in 2008. In the same year Sharma [37] defined the fractional integration and fractional differentiation of the M-Series.

In 2009, Jumarie [20] found the work on table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. In the same year Sharma and Jain [39] provided a note on a generalized M-series as a special functions of fractional calculus.

Jain, Saxena, Mishra [63] have applied fractional calculus approach which based on the Convolution theorem for fractional Fourier transform and Rivero, Rodriguez-Germa, Trujillo and Velasco [40] have applied fractional operators on some special functions in 2010. In the same year Kiryakova [74] defined the special functions of fractional calculus operators of some basic functions.

In 2011, Machado, Kiryakova, and Mainardi [24] have given the recent history of fractional calculus.

In 2012, Prabhakaran and Rao [9] obtained the Saigo operators of fractional integration of hyper geometric functions and Agarwal [53] defined further results on fractional calculus of Saigo operators in the same year.

Singh and Rawat [14] have provided the integral involving generalized Mittag-Leffler Function in 2013..

In 2013, Machado, Galhona and Trujillo [23] have given the science metrics on fractional calculus development since 1966. Also they have discussed progress of fractional calculus area last fifty years, conference devoted to fractional calculus and special issues in journals, Books devoted to fractional calculus.

In 2014, Kareem [54] developed the numerical methods for fractional differential equations.

In 2015, Losada and Nieto [26] obtained the properties of a new fractional derivative without singular kernel and Ali, Sharma and Jain [45] given an application of fractional calculus in RLC circuit in the same year.

In 2016, Sharma [38] defined computable extensions of advanced fractional kinetic equation and a class of levy-type probabilities.

2. NOTEWORTY CONTRIBUTION IN THE FIELD OF PROPOSED WORK:-

Several renowned mathematicians have enriched this field by their contribution notably Leibniz, Abel, Euler, Liouville, Riemann, Weyl, Lacroix, Fourier, Grunwald and Letnikov etc.

(1)Title of the paper: "Fractional derivatives for Kober operators and statistical densities in The real matrix variate cases"

Name of Journal and Year: Fractional calculus and application (2015)

Author: A. M. Mathai [7]

In this research paper, Author studied the Fractional integrals, Fractional derivatives and Fractional differential equations in the real scalar variable cases have obtained many applications. Author represented the fractional integrals in the real and complex matrix-variate cases especially fractional derivatives involving Kober operators of the first and second kind in the matrix-variate case are also discussed.

This paper Author given the densities $g_2(u_2)$ and $g_1(u_1)$ then it is shown in [6] that

$$g_2(u_2) = \frac{\Gamma p(\alpha+\gamma+\frac{p+1}{2})}{\Gamma p(\gamma+\frac{p+1}{2})} K_{2,w}^{-\alpha}$$

$$g_1(u_1) = \frac{\Gamma p(\alpha+\gamma)}{\Gamma p(\gamma)} K_{1,w}^{-\alpha}$$

Where $K_{2,w}^{-\alpha}$ is the Kober fractional integral operator of order α of second kind is defined as:

$K_{2,w}^{-\alpha} f = \frac{[\det(w)]^{\rho}}{\Gamma p(\alpha)} \int_{v>u} [\det(v)]^{-\rho-\alpha} [\det(v-w)]^{-\alpha-\frac{p+1}{2}} f(v)dv$ For $\Re(\alpha) > \frac{p-1}{2}$

. $K_{1,w}^{-\alpha}$ is the Kober fractional integral operator of order α of first kind is defined as:

$$K_{1,w}^{-\alpha} f = \frac{[\det(w)]^{-\rho-\alpha}}{\Gamma p(\alpha)} \int_{v<w} [\det(v)]^{\rho} [\det(w-v)]^{\alpha-\frac{p+1}{2}} f(v)dv$$

The statistical density by multiplying with a constant

$$f(v) = \frac{\Gamma p(\rho)}{\Gamma p(\frac{p+1}{2}) \Gamma p(\rho-\frac{p+1}{2})} [\det(v)]^{\frac{p+1}{2}-\frac{\rho+1}{2}} [\det(I+v)]^{-\rho}, v > 0$$

Which is a type-2 matrix-variate beta density with parameter $(\frac{p+1}{2}, \rho-\frac{p+1}{2})$

$$f(v) = \frac{1}{\Gamma p(\rho)} [\det(v)]^{\rho-\frac{p+1}{2}} [\det(I-w)]^{\frac{p+1}{2}-\frac{\rho+1}{2}}$$

Hence $\frac{\Gamma p\left(\rho+\frac{p+1}{2}\right)}{\Gamma p\left(\frac{p+1}{2}\right)} f(v)$ is a statistical density, which is a type-1 matrix-variate beta density with parameters $(\rho, \frac{p+1}{2})$.

(2) Title of the paper: "A new definition of fractional derivative without singular kernel"

Name of Journal and Year: Progress in fractional differentiation and applications (2015)

Author: Michele Caputo and Mauro Fabrizio [43]

In this research paper, Authors represented a new definition of fractional derivative without singular kernel for the temporal and spatial variable. The first work on the time variables; thus it is suitable use the Laplace transform and second related to the spatial variables by a non-local fractional derivative for which it is more convenient to work with the Fourier transform.

We know that the usual Caputo fractional time derivative of order α is given by

$$D_t^{(\alpha)} f(\theta) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f(\theta)}{(t-\theta)^{\alpha}} d\theta$$

With $\alpha \epsilon [0,1]$ and $a \in [-\infty, t)$, f$\in H^1(a,b), b > a$ changing the kernel (t-θ)$^{-\alpha}$ with the function exp($\frac{-\alpha}{1-\alpha} t$) and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{(1-\alpha)}$.

Obtain new definition of fractional time derivative

$D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f(\theta) \exp\left[\frac{-\alpha(t-\theta)}{1-\alpha}\right] d\theta \ldots$ (1)

Where M (α) is a normalization function such that M (0) = M (1) =1. When $f(t)$ is constant, as in the usual Caputo fractional derivative, but contrary to the usual Caputo fractional time derivative. The kernel does not have singularity for $t = \theta$

(3) Title of the paper: "A new definition of fractional derivative"

Name of Journal and Year: Computational and Applied Mathematics (2014)

Author: R. Khalil, M. Al Horani, A. Yousef and M. Sababheh [66]

In this paper, Authors given a new definition of fractional derivative and fractional integral for $0 \leq \alpha < 1$ coincide with the classical definition on polynomials and if $\alpha = 1$ coincides with the classical definition of first derivative and last authors represented an applications to fractional differential equations.

Def: - authors given a function$f$: [0,∞) $\rightarrow R$ then the conformable fractional derivative of f of order a is defined by

$T_a(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-a}) - f(t)}{\epsilon}$ For all t>0, $a \in (0,1)$

If f is a-differentiable in some (0, b), b>0 and $\lim_{t \to 0^+} f^a(t)$ exists, then define

$$f^{(a)}(0) = \lim_{t \to 0^+} f^a(t)$$

Def: - Let $a \in (n, n+1]$ and f be an n-differentiable at t, where t>0. Then the conformable fractional derivative of f of order a is defined as

$$T_a(f)(t) = \lim_{\epsilon \to 0} \frac{f^{([a]-1)}\left(t+\epsilon\, t^{([a]-a)}\right) - f^{([a]-1)}(t)}{\epsilon}$$

Where $[a]$ is the smallest integer greater than or equal to a and easily show that $T_a(f)(t) = t^{([a]-a)} f^{[a]}(t)$

Where $a \in (n, n+1)$ and f is $(n+1)$ differentiable at $t > 0$

Def: - definition for the a-differentiable integral of a function f, for b≥0 given that

$$I_\alpha^b(f)(t) = I_1^b(t^{a-1} f) = \int_b^t \frac{f(x)}{x^{1-a}} dx$$

Where the integral is the usual Riemann improper integral and $a \in (0,1)$

(4) Title of the paper: "Applications of fractional Fourier transform convolution theorem in Filtering"

Name of Journal and Year: International Journal of information engineering (2013)

Author: Rajshree Mishree, Rajiv Saxena and Renu Jain [55]

In thisresearch paper, Authors studied the fractional Fourier transform is a generalization of the classical Fourier transform and the applications of fractional Fourier transform in different areas like signal processing, optics and quantum mechanics.

The fractional Fourier transform with fractional angle α of a function f in $L^1(R)$ is defined

$$f_\alpha(u) = \int_{-\infty}^{\infty} f(t) k_\alpha(t,u) dt$$

$k_\alpha(t,u)$ is defined

$$k_\alpha(t,u) = \begin{cases} \frac{c_2}{\sqrt{2\pi}} \exp\left(j\left\{\frac{a_1 t^2 + a_1 u^2 - 2b_1 ut}{2}\right\}\right) & if\ a \neq 0, \frac{\pi}{2}, \pi \\ \delta(t-u) & if\ \alpha = 2m\pi \\ \delta(t+u) & if\ \alpha = (2m+1)\pi \\ \frac{1}{\sqrt{2\pi}} e^{-jut} & if\ \alpha = \frac{\pi}{2} \end{cases}$$

Where $k_\alpha(t,u)$ is kernel of the transform and $a_1 = cot\alpha\ ; b_1 = cosec\ \alpha\ ;\ c_1 = \sqrt{1} - Jcot\alpha$

In this paper, Authors conclude a low pass filter, designed by using convolution theorem. The design of filter, Blackman window function is used. They observe the proposed Fractional Fourier Transform domain filtering is far better than the classical Fourier transform domain filtering or even time domain filtering. Authors obtained the proposed convolution structure is a generalization of the existing convolution structure of Fourier transform to fractional Fourier transform and filtering by using proposed definition of fractional Fourier transform convolution shows superior performance than the time/frequency domain filtering in terms of denoising the signal.

(5) Title of the paper: "On the generalized Mittag-Leffler and its applications in a fractional Telegraph equation"

Name of Journal and Year: Mathematical Physics, Analysis and Geometry (2012)

Author: Rubens Fiqueiredo Camargo, Edmundo Capelas Olivera and Jayme Vaz [67]

In this paper, Authors introduced some integral transform associated with this generalized Mittag-Leffler function and this paper as a particular case some recent result are recovered. In this paper authors studied of fractional order differential and integral equations is a important role play of the classical Mittag-Leffler function, involving one and two parameter.

Mittag-Leffler function of two parameter

$$E^1_{\alpha,\beta}(x) = E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}$$

$\rho = 1, \beta = 1$ then the classical Mittag-Leffler function

$$E_{\alpha,1}^{1}(x) = E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}$$

Then authors mention the relation [56]

$$E_{\alpha,\beta}^{\gamma}(x) = \frac{1}{\Gamma\gamma} 1^{\psi_1}[x \mid {}^{(\gamma,1)}_{(\beta,\alpha)} \mid] = \frac{1}{\Gamma\delta} H_{1,z}^{1,1}\left[-x \mid {}^{(1-\delta,1)}_{(0,1),(1-\beta,\alpha)}\right]$$

Where $1^{\psi_1}\left[x \mid {}^{(a,b)}_{(p,q)}\right]$ is the Wright function which was presented as solution of the fractional equation associated with the renewal process of Mittag-Leffler and Wright type [18] and this function generalizes the classical confluent hyper geometric function $1^{F_1}(a; c; x)$

We can write

$$E_{1,\beta}^{\delta}(x) = \frac{1}{\Gamma\delta} 1^{F_1}(\beta; \delta; x) \quad for\ Re(\beta) > 0, and\ Re(\delta) > 0$$

Conclude that the generalized Mittag-Leffler function was introduced using the Juxtaposition of integral transforms, namely the Laplace transform in the time variable and Fourier transform in space variable. Authors discussed the fractional Telegraph equation and obtain the solution in terms of Fox's x function. After that author discussed in this paper the Green function associated with the fractional Telegraph equation and obtained a possible relation between Green function associated with the diffusion equation and the wave equation.

(6) Title of the paper: "On the fractional signals and systems"

Name of Journal and Year: Signal Processing (2011)

Author: Richard Magin, Manuel D. Ortiqueira, Igor Podlubny and Juan Trujillo [65]

In this research paper, Authors represented a fractional calculus and its applications from the signal processing. An interesting feature of the systems is the causality that the fractional derivative. Authors presented the main properties of the derivatives. A brief and general study of the fractional linear systems by showing how to compute the impulse, step and frequency responses and how to test the stability, how to insert the initial conditions. Authors performed the input-output numerical computations in general nonlinear systems (Diethelm's method based on quadrature Lubich's difference methods, Adams-Bashforth-Moulton method,

Adomian's decomposition method, Homotopy-perturbation method) and some biomedical applications are discussed

Authors introduced the modification of the fractional derivative by the limit of the fractional derivative by the limit of the fractional incremental ratio

D_θ^α(x)=$e^{-j\theta\alpha}\lim_{|h|}\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x-kh)}{|h|^\alpha}$ where $\binom{\alpha}{k}$ stands for the binomial coefficients and h=$|h|e^{j\theta}$ is a complex number with $\theta\epsilon(-\pi,\pi]$. This definition valid for any order, real and complex [25].

Authors discussed the Riemann-Liouville, Caputo derivatives and defined input-output numerical computation in general nonlinear systems

(1) Adomian's decomposition method – this method was firstly use to obtained the approximate solution of linear or nonlinear differential equations [47] but authors specially finding the solution of the Abel-Volterra integral equation

$y(m)$=$\int_0^m(m-\tau)^{\alpha-1}f[\tau,y(0)u(m)$

Solution as

$$y(m)=g(m)+\frac{1}{\Gamma\alpha}\int_0^m(m-\tau)^{\alpha-1}\sum_{i=0}^{\infty}f^{A_i}(\tau)d\tau$$

Where the $f^{A_i}(m)$ are the Adomian polynomials and $g(m)$. Authors include the input and initial condition term. The explicit scheme of Adomian's decomposition method can be written as

$$y_0(m)=g(m),\qquad y_{i+1}(m)=\frac{1}{\Gamma\alpha}\int_0^m(m-\tau)^{\alpha-1}f^{A_i}(\tau)d\tau\quad i=0,1,2\ldots..$$

Where$f^{A_i}(m)=[\frac{1}{i!}\frac{d^i}{d\lambda^i}f(m,\sum_{j=0}^{i}\lambda^j y_j)]_{\lambda=0}$

(2) Homotopy-Perturbation method- we construct the following homotopy see [29,30] as

$D^{\alpha_i}y_i=pf_i(m,y_1,y_2\ldots..y_n)$ --- (a)

Where i=1, 2, 3---n and p is an embedeling parameter that changes from zero to unity [51]

If p=0 we will obtain the linear equation $D^{\alpha_i}y_i$=0

And if p=1 the homotopy equation (a) turns out to be the original system given in

$D^{\alpha_n}y_n(m)=f_1(m,y_1,y_2\ldots..y_n)$

Solution of system (a) can be expanded using the parameter p

$$y_i(m) = y_{i0} + py_{i1} + p^2 y_{i2} + \cdots (b)$$

The value of y_i put in the equation (a) and authors obtain the linear equation of the form [51]

$$p^0 : D^{\alpha_i} y_{i0} = 0$$

$$p^1 : D^{\alpha_i} y_{i1} = f_{i1}(m, y_{10}, y_{20} \ldots\ldots y_{n0})$$

$$p^3 : D^{\alpha_i} y_{i3} = f_{i3}(m, y_{10}, y_{20} \ldots\ldots y_{n0}, y_{11}, y_{21} \ldots\ldots, y_{n1}, y_{12}, y_{22} \ldots\ldots, y_{n2})$$

:

: …… (c)

:

Where the function y_{i1}, y_{i2} ... satisfy the following equation

$$f_i(m, y_{10}, py_{11} + p^2 y_{12} + \cdots, y_{n0} + py_{n1} + p^2 y_{n2} + \cdots)$$
$$= f_{i1}(m, y_{10}, y_{20}..y_{n0}) + pf_{i2}(m, y_{10}, y_{20} \ldots\ldots, y_{n0}, y_{11}, y_{21} \ldots y_{n1})$$
$$+ p^2 f_{i3}(m, y_{10}, y_{20} \ldots\ldots, y_{n0}, y_{11}, y_{21} \ldots y_{n1}, y_{12}, y_{22} \ldots\ldots\ldots y_{n2}) + \cdots (d)$$

The linear equations can be easily solved by applying the operator J^{α_i}, i.e. The inverse of the fractional operator D^{α_i} .The component y_{ik}, k=0, 1, 2…

The HPM solution can be determined, if p=1 in equation (d) determine the entire HPM series solution

$$y_i(m) = \sum_{k=0}^{\infty} y_{ik}(m)$$

The HPM series solution equation (c) can be approximated by the N-term truncated series

$$\phi_{iN}(m) = \sum_{k=0}^{N-1} y_{ik}(m)$$

(7) Title of the paper: "An alternative method for solving a certain class of fractional kinetic equations"

Name of Journal and Year: In Proceedings of the Third UN/ESA/NASA Workshop On The International Heliophysical 2007 and Basic

Space Science (2010)

Author: R.K.Saxena, A.M.Mathai and H.J.Haubold [61]

R.K.Saxena, A.M.Mathai and H.J.Haubold in above mentioned paper have given the solution of fractional kinetic equations are solved earlier by Haubold and Mathai [21], Saxena et.al [58,59,60] and recently given Saxena and Kalla [58]. Authors represented an alternative simple method for deriving the solution of the generalized forms of the fractional kinetic equations solved for deriving the solution of the generalized forms of the fractional kinetic equations solved by the aforesaid authors and Nonnenmacher and Metzler [71]. Applications of Riemann-Liouville fractional calculus operator and its properties, that the solution of given fractional kinetic equation can be obtain in a straight forward manner.

The Riemann-Liouville operators of fractional calculus are defined in books by Miller Ross [33], Oldham and Spanier [32], Podlubny [22] and Kilbas, Srivastava and Trujillo [5] as:

$${}_{\alpha}D_x^{-v}N(x)=\frac{1}{\Gamma v}\int_{\alpha}^{x}(x-u)^{v-1}\,N(u)du,\ Re(v)>0,\ x>a\ldots.\ (1)$$

With ${}_{a}D_x^{\mu-n}N(x)=N(x)\ {}_{\alpha}D_x^{0}N(x)=\frac{d^n}{dx^n}\left(({}_{\alpha}D_x^{\mu-n}N(x)\right),Re(\mu)>0,n-\mu>0$

By virtue of the equation (1) it is show that ${}_{a}D_x^{-v}(x-a)^{\rho-1}=\frac{\gamma(\rho)}{\Gamma(\rho+v)}(x-a)^{\rho+v-1}\ldots(2)$

[see(22)] , Authors have ${}_{a}D_x^{v}(x-a)^{\rho-1}=\frac{\gamma(\rho)}{\Gamma(\rho-v)}(x-a)^{\rho-v-1}$

Where $Re(v)>0,\ Re(\rho)>0,x>a\ and\ \rho=1$

Then ${}_{a}D_x^{v}1=\frac{1}{\Gamma(1-v)}(x-a)^{-v},\ x>a\,;v\neq 1,2\ldots$

Authors obtained the result in the theory of fractional calculus and fractional derivative of a constant is not zero.

In the next section authors discussed the derivation of the fractional kinetic equation and its solution

The standard kinetic equation $\frac{d}{dx}N_i(x) = -C_iN_i(x),\ (C_i > 0)$….. (3)

Authors derived the fractional kinetic equation $N(x) - N_a = -C^v\ {}_aD_x^{-v}N(x)$ … (4)

Using the equation (2) and obtained the solution of fractional kinetic equation (4)

$N(t) = N_aE_v[\text{-}C^v(x-a)^v], x > a$

(8) Title of the paper: "A new operational matrix for solving fractional-order differential equations"

Name of Journal and Year: Computers and Mathematics with Applications (2010)

Author: Abbas Saadatmani and Mehdi Dehghan [1]

In this paper, Authors studied the numerical solution of a class of fractional differential equations and a general formation for the Legendre operational matrix of fractional derivative has been derived in the Caputo sense. Authors aim in this paper Legendre series together weyl the Legendre operational matrix of fractional derivative are used for numerical integration of fractional differential equations, using this technique then it reduce the problem solving a system of algebraic equations. This method applied to solve two types of fractional differential equations, linear and nonlinear

In this section authors given the Caputo definition of the fractional order derivative is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}}, \qquad n-1 < \alpha \le n, \qquad n \in N$$

For the Caputo derivative [31]

$$D^\alpha C = 0$$

$$D^\alpha x^\beta = \begin{Bmatrix} 0 & for\ \beta \in N_0, \beta < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha} & for\ \beta \in N_0, \beta < [\alpha] \end{Bmatrix}$$

The Caputo fractional differentiation is a linear operation

$$D^\alpha(\lambda f(x) + \mu g(x) = \lambda D^\alpha f(x) + \mu D^\alpha g(x)$$

In the next section authors discussed the generalized Legendre operational matrix to fractional calculus and next section author studied the operational matrix of fractional derivative, using the solution of multi order fractional differential equation and another section authors given some example and solution

5. OBJECTIVES OF THE PROPOSED RESEARCH:-

Fractional calculus has become the new most existing tool for dealing with various diverse problems in mathematical physics, science, engineering and statistics. Due to its diversified applications there is a lot of scope of research in this area. There is a possibility of improvement/extension of existing results and scope for the establishment of new results. There is a huge scope of applications to this special class of mathematics. In view of this following are the research objective:

(a) To make extensive and comparative study of fractional calculus operators including Riemann-Liouville, Caputo, Grunwald-Letnikov, Weyl, Kober operators, Mittag-Leffler function.
(b) To extend/generalize existing results.
(c) To produce novel results to enrich the theory of fractional calculus.
(d) To derive a new results by using new definitions of fractional derivative.
(e) To apply novel/existing results to produce some new applications.

6. METHODOLOGY:-

There are various steps in research process; each step of the proposed research work is given below-

(1) Literature Review:
. Make a list of Journals related to Fractional Calculus.
. Select the concerned articles.
. Organize sorted articles according to Serial Number, Year, Authors, Title, Abstract, Journal and Reference.

(2) Step 1: Research gaps and problems are determined on the basis of the highlighted issues
Step 2: Objectives are formulated on the basis of research gaps and problems identified. In the proposed methodology, planned objectives are as follows:

. Generalization/extension of some existing results by using fractional integral operators.

. Establishment of some new results by using fractional integral operators.

. Establishment of some new results by using some new definitions a fractional derivative.

. Applying a new/existing results to produce new applications.

Research Methodology Steps:-

1. Problem → Literature Review
2. Research Gaps
3. Objectives→ Now either
   (a) Extension / Improvement in existing results
   (b) Establishment of new results
4. Applications of fractional calculus in science, engineering and statistics in general sense, in particular sense finding applications of fractional calculus in kinetic equations, physical equations. In engineering finding the applications of fractional calculus in electrical, mechanical, automobile, civil engineering and statistics in levy type probabilities.
5. Analysis of the result
6. Comparisons with existing results/applications
7. Conclusions made on the basis of comparison
8. End

7. EXPECTED OUTCOME OF THE PROPOSED WORK:-

A calculus of arbitrary order that is fractional calculus is now widely applicable in almost every area of applied mathematics, science and engineering. Due to its variety of applications and scope of its theoretical development, it is expected that in the proposed research work some new results will be established and also some existing results will be extended. It is also expected that some new applications will be obtained specifically in the field of science and engineering. It is also expected that mathematical modeling that can be treated with the fractional calculus operators will also be used to represent certain physical phenomena which will generate useful applications.

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